Concurrent games with symmetry

Intership at the Computer Laboratory supervised by Glynn Winskel and Pierre Clairambault

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Introduction

- Concurrent games: game semantics based on event structures, for concurrent programs
- Game semantics: execution of a program = interaction between two entities, a **program** respecting some **specification** = a **winning strategy** over a **game**
- As of [**?**], concurrent games do not handle duplication: eg. $S = \lambda xyz.\ x\ z\ (y\ z)$
- Goal of the internship: add duplication to the framework

Overview

Duplication: linear logic provides a good framework to deal with duplication

- *Concurrent games of [**?**]*: only a model the linear fragment (IMLL) ($\rightarrow$ linear λ-calculus)
- *Our goal*: add an **exponential** to have IMELL ($\rightarrow$ simply typed λ-calculus)

Exponential: construction on games: $A \mapsto !A$ "A repeated infinitely many times" (similar to the free comonoid construction)

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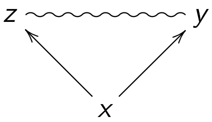
Event structures

- Event structures have been invented to give a denotational model of message-passing style concurrency (eg. CCS)
- An event structure is a set of *events* related by two notions:
  - *causality*: an order on events
  - *consistency*: a predicate on set of events, “Is X consistent?”



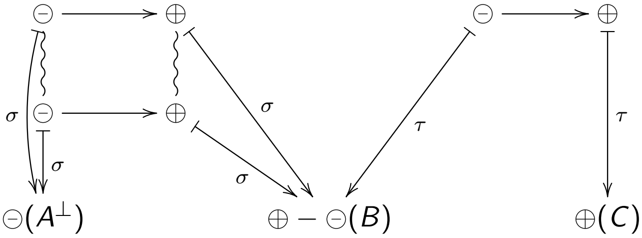
- *configuration*: possible “history” (downward closed and consistent set of events)

Games and prestrategies

- *Game*: event structure E *with polarities*
- *Operations on games*: dual ($A^\perp$), parallel composition $A||B$
- *Prestrategies on a game* A: a map of event structures preserving polarities $\sigma : S \to A$, ie.
  - σ preserves the notion of configuration
  - σ is locally injective (inside a configuration)
- *Prestrategies from* A *to* B: a prestrategy on $A^\perp||B$

Composition of prestrategies

Given $\sigma : S \to A^{\perp}||B$ and $\tau : T \to B^{\perp}||C$ we build $\tau \odot \sigma : S \odot T \to A^{\perp}||C$.
Example on $A = B = C = \oplus$.



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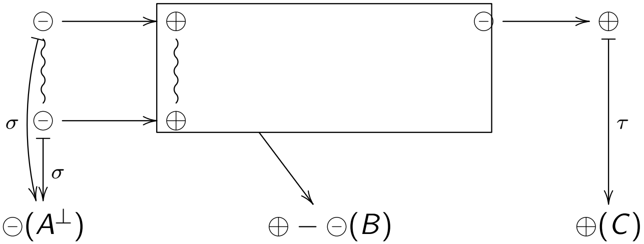

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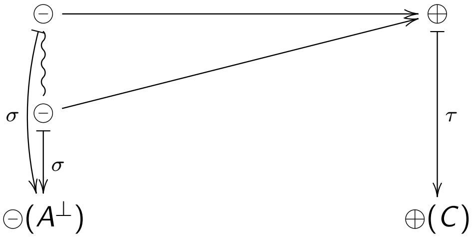

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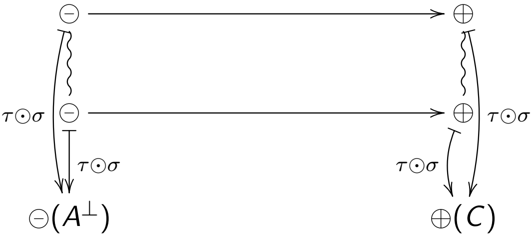
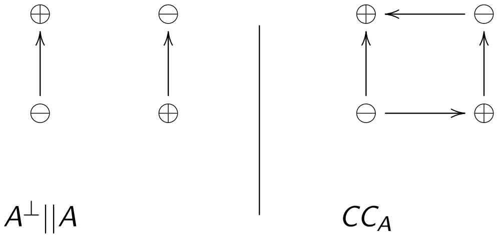


Copy-Cat

- Identity for $\odot$ on a game A ? **Copy-Cat**: $\gamma_A : CC_A \to A^{\perp}||A$



- γ_A intuitively waits for the opponent to make move in A or $A^{\perp}$ and then plays the corresponding move in the other copy.
- **Idempotence** up to isomorphism: $\gamma_A \odot \gamma_A \cong \gamma_A$

The bicategory of games

Strategy: Prestrategy $\sigma : S \to A^{\perp}||B$ such that
$\sigma \odot \gamma_A \cong \gamma_B \odot \sigma \cong \sigma$.

- We have a bicategory of games and strategies
- This category is able to model the linear λ-calculus: $A \to B$ is interpreted by $A^{\perp}||B$
- Linearity arises because we can only play (= consume) A once.

The problem

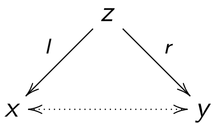
- Given a game A, we want a game $?A$ which represent "as many copies of A as we like" such that $A \mapsto ?A$ can be seen as an **exponential** (note: ? operator is the dual of !)
- Natural candidate: $A \times \mathbb{N}$
- No link between the copies of $A \Rightarrow$ monadic laws fail to hold
- Need a weaker notion of equivalence on strategies (than $\cong$) up to which the monadic laws hold
- Introduction of the notion of *symmetry*:
  1. Defines an equivalence on configurations: $x \simeq y$ meaning that x and y are interchangeable
  2. Yields an equivalence on maps: $f \sim g$ when $fx \simeq gx$ for any configuration x

Definition by bisimulation

- A symmetry on A is an event structure $\widetilde{A}$ behaving like a bisimulation on A

$$A \xleftarrow[l]{} \widetilde{A} \xrightarrow[r]{} A$$

- $x \simeq y$ when there exists a configuration z in the symmetry such that

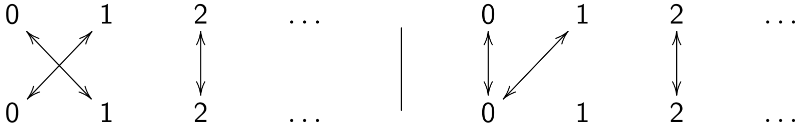


- there is a category of event structures with symmetry (ess), maps of ess $f : A \to B$ are maps that preserves $\simeq$.

The monad

We keep $?A = A \times \mathbb{N}$. Need a **symmetry** on $?A$ that identify the different copies.

- forall $a \in A$, (a, i) and (a, j) are the same so we need a witness $i \leftrightarrow^a j$ in the symmetry
  $\Rightarrow \widetilde{?A} = A \times \mathbb{N}^2$ the set of such arrows.
- *Consistency* on $\widetilde{?A}$:



Monadic operators (laws hold up to $\sim$):

- $\eta : A \rightarrow ?A, a \mapsto (0, a)$
- $\epsilon : ??A \rightarrow ?A, (a, i, j) \mapsto (a, \langle i, j \rangle)$

The bicategory of games with symmetry, *SGames*

- Goal: having a bicategory whose objects are event structures with symmetry and polarities (essp)
- Problem: depending on A, CC_A may not have a symmetry $\Rightarrow$ stronger notion of games
- *Game with symmetry*: an essp A such as A and $\widetilde{A}$ are race-free (no irreversible choice between events of different polarities)
- *Symmetric strategies from A to B*: map of essps $\sigma : S \to A^\perp || B$ such as σ and $\widetilde{\sigma}$ are strategies
- With these definitions, we have
  - composition: $\widetilde{\tau \odot \sigma}$ is simply $\widetilde{\tau} \odot \widetilde{\sigma}$
  - identity: $\widetilde{\gamma_A} = \gamma_{\widetilde{A}}$.

Theorem

Games with symmetry and symmetric strategies form a bicategory SGames (for $\cong$)

Equivalence on strategies

- What is the translation of $\sim$ (defined in the category of essps) in *SGames* ?
- $\sigma : S \to A^{\perp} || B \simeq \tau : T \to A^{\perp} || B$ when there exists a family $\mathbb{S}$ of bijections between configurations of S and T such that
  - $\mathbb{S}$ has the same bisimulation properties as for the def. of symmetry
  - Stability by action of the symmetry on S (on the left) and on T (on the right)
  - If $\theta : x \cong y$ and $\theta' : x' \cong y$ then $\theta'^{-1} \circ \theta : x' \cong x$ is in $\widetilde{S}$ (same thing on T)

Lemma

$\simeq$ *is a congruence containing* $\cong$.

From event structures to games

- How to lift the monad components from essps to games ? From $f : A \to B$ to $\sigma(f) : S_f \to A^{\perp}||B$.
- From event structures to games: S_f is $A^{\perp}||B$ by adding links: $\bar{a} \to f(a)$ when a is positive and $f(a) \to \bar{a}$ otherwise.
- To make the construction work with symmetry we need to add hypothesis on f:

Theorem

There exists a functor $\sigma : E \to SGames$ from the category E of event structures with symmetry and rigid, injective and surjective maps preserving equivalence (meaning that $f \sim g$ entails $\sigma(f) \simeq \sigma(g)$)

Polarisation

Problem: The symmetry on $?A$ is not always race-free $\Rightarrow$ **polarisation**.

- A game A is $+$-polarised if all minimal events are positive (and negative for $-$ polarised).
- We have two dual subcategories $SGames^-$ (and $SGames^+$) of games that are $-$-polarised (or $+$-polarised) with strategies negatively polarised (opponent starts)
- $E^{+/-}$: category of polarised essps, if $F : E^{+/-} \to E^{+/-}$ preserves the structure, then F induces endofunctors in $SGames^-$ and $SGames^+$.

Theorem

Let $F, G : E^{+/-} \to E^{+/-}$ be functors preserving the structure, and $\eta : F \Rightarrow G$ be a natural isomorphism. Then $\sigma(h) : F^+ \Rightarrow G^+$ is a natural isomorphism (same for F^- and G^-)

A linear exponential comonad.

- We have a monad ? on $SGames^+$, by duality a **comonad** ! $SGames^-$.
- A few checks lead to

Theorem

! is an exponential comonad. It means that the category $Games_\lambda$:

- *whose objects are games with symmetry negatively polarised*
- *whose maps from A to B are polarised strategy with symmetry from $!A$ to B*

is a model of simply-typed λ-calculus (CCC)

Conclusion

- We have seen how to add duplication inside concurrent games.
- This way is very natural to traditional game semantics (polarisation).
- But concurrent games is actually a model of MLL (the bicategory is self-dual).
  $\Rightarrow$ we would like a model of MELL, without polarisation.
- This may be possible by changing completely composition and copy-cat to better cope with symmetry.