Symmetry in Concurrent Games

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Motivation and context

Motivation:

1. What? Develop and extend the "truly concurrent" approach to game semantics based on partial orders, to allow for **replication** through **symmetry**.
2. Why? Obtain a finer representation of programs and their execution in a more elegant mathematical framework.
3. How? Interpret strategies as event structures to focus on **causality**.

Related work:

- Notions of deterministic concurrent strategies: Abramsky, Melliès, Mimram, Faggian, Piccolo; Orbital games: Melliès
- Strategies as presheaves: Hirschowitz, Pous
- Non-deterministic concurrent strategies as event structures: Rideau, Winskel.
  → We will work with this framework.

I. Concurrent Games

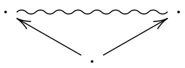
Event structures and their maps

Definition (Event structure)

An event structure E is a set of event E along with

- an order $\leqslant_E$ (**causality**)
- a set $\mathrm{Con}_E \subseteq \mathscr{P}_f(E)$ (**consistency**)

satisfying some axioms.



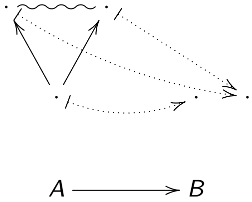
Set of **configurations** of E:

$$\mathscr{C}(E) = \{x \subseteq E | x \in \mathrm{Con}_E \ \&\ x \text{ down-closed}\}$$

Definition (Maps of event structures)

A map $f : A \rightarrow B$ is a function on events satisfying:

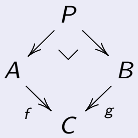
- Preservation of configurations: $x \in \mathscr{C}(A) \Rightarrow f\, x \in \mathscr{C}(B)$
- Local injectivity: If $x \in \mathscr{C}(A)$ then f defines a bijection $x \overset{f}{\cong} fx$



Pullbacks in event structures

Proposition

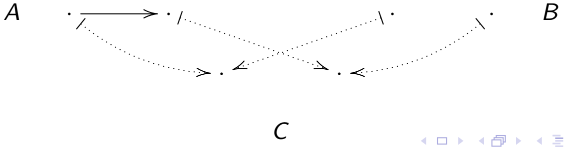
The category of event structures has all pullbacks:



Configurations of the pullback are given by composite bijections:

$$\mathscr{C}(A) \ni x \overset{f}{\cong} fx = gy \overset{g}{\cong} y \in \mathscr{C}(B)$$

*inducing no causal loops (***secured bijections***)*



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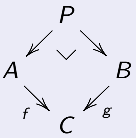

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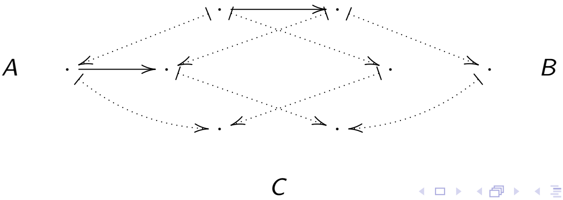

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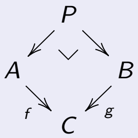

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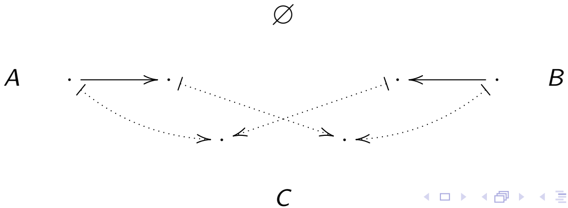
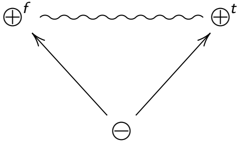


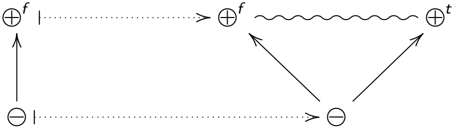
Games and pre-strategies

A **game** is an event structure E where each event has a polarity ($\oplus$ or $\ominus$)

$\mathbb{B} =$



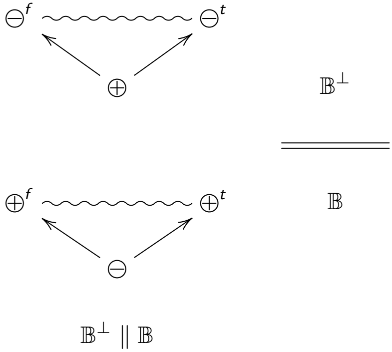
A **pre-strategy** on a game A is a map $\sigma : S \rightarrow A$



$$\text{false} : F \longrightarrow \mathbb{B}$$

Pre-strategies from one game to the other

Pre-strategies from A **to** B are pre-strategies on the game $A^{\perp} \| B$ where $\|$ is parallel composition – no conflict or caulities between A and B.



Notation : $\sigma : A \multimap B$

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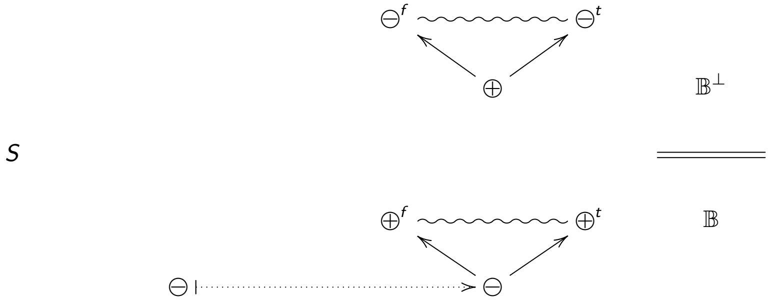

$$\text{neg} : S \longrightarrow \mathbb{B}^{\perp} \parallel \mathbb{B}$$

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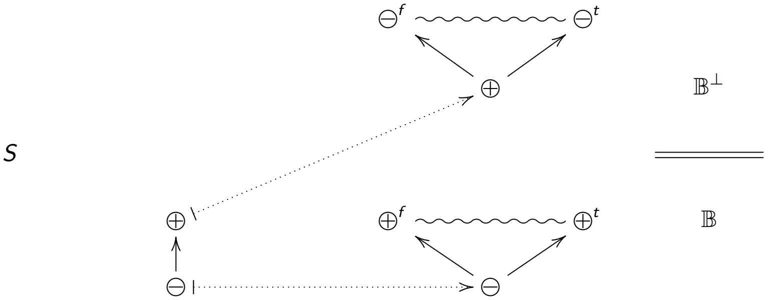

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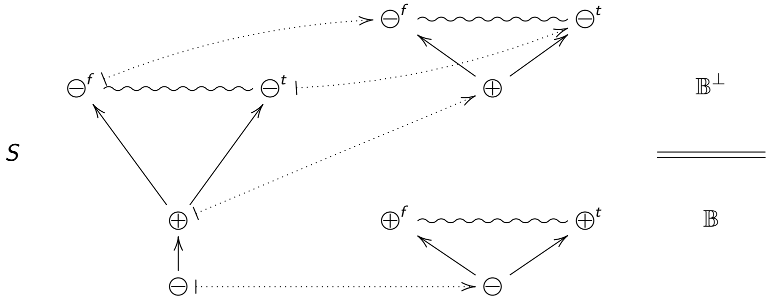

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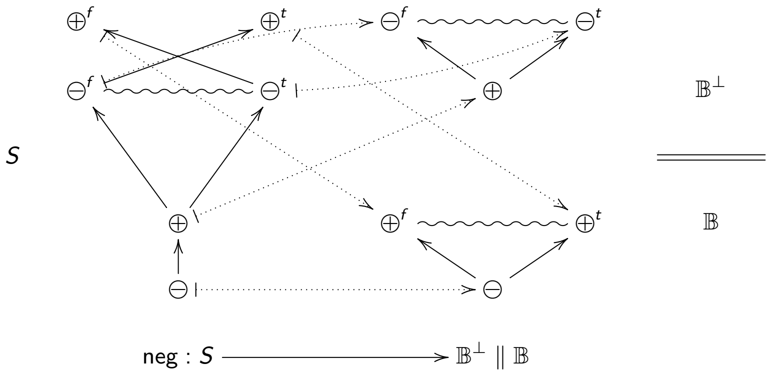
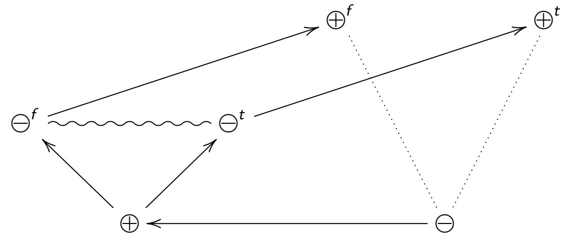


Notation : $\sigma : A \rightarrow\!\!\!\!+\!\!\!> B$

Towards a category : Identity

Copycat strategy: $\gamma_A : \mathbb{CC}_A \to A^\perp \| A$, forwards negative moves on one side to the other side.

$\mathbb{CC}_\mathbb{B} =$



Configurations of copycat: pair of configurations (x, y) of A such that

- $x \cap y \subseteq^- y$
- $x \cap y \subseteq^+ x$

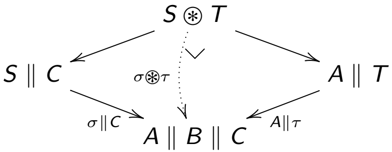
which we write $y \sqsubseteq_A x$ (Scott order on A)

$$\mathscr{C}(\mathbb{CC}_A) \cong \{(x, y) \in \mathscr{C}(A)^2 \mid y \sqsubseteq x\}$$

Composition of pre-strategies

composition = interaction + hiding.

Interaction of pre-strategies $\sigma : S \to A^{\perp} \parallel B$ and $\tau : T \to B^{\perp} \parallel C$ via a **pullback**:



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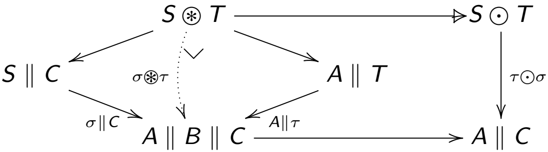
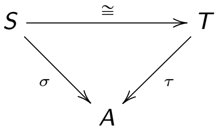


A bicategory of games

- What notion of equivalence for strategies? **Isomorphism**:



- Not all strategies behave well with respect to copycat up to isomorphim. Only the **innocent** and **receptive** ones.

Theorem (Rideau, Winskel)

A pre-strategy $\sigma : A \rightarrow\!\!\!\!+\!\!\!> B$ *is innocent and receptive iff it satisfies* $\gamma_B \odot \sigma \odot \gamma_A \cong \sigma$

- Defining strategy to mean innocent and receptive pre-strategy, we have a bicategory of games and strategies.
- Goal: to add symmetry to the framework to allow for finer equivalences — to model replication for instance.

II. Event structures with symmetry

Event structures with symmetry

Definition (Event structure with symmetry)

An event structure with symmetry $\mathcal{A}$ is given by a span in the category of event structures:

$$A \xleftarrow{\;l_A\;} \tilde{A} \xrightarrow{\;r_A\;} A$$

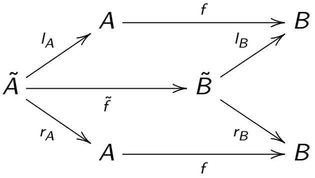
where l_E, r_E are open (ie. have a **bisimulation**-like lifting property), jointly monic and form an equivalence relation.

More concretely, an event structure with symmetry can be given by a pair $\mathcal{A} = (A, \mathbb{S}_A)$ where $\mathbb{S}_A$ is a set of bijections between configurations of A

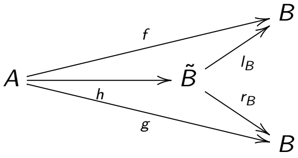
- that contains **identities** and is stable under **inverse** and **composition**
- if $x \overset{\theta}{\cong}_A y \in \mathbb{S}_A$ then any **extension** or **restriction** of x induces a restriction or an extension of θ.

Maps of event structures with symmetry

A map $f : \mathcal{A} \to \mathcal{B}$ is given by two maps $(f : A \to B, \tilde{f} : \tilde{A} \to \tilde{B})$ making the following commute:



For $f, g : \mathcal{A} \to \mathcal{B}$, we write $f \sim g$ iff there exists a map $h : A \to \tilde{B}$ such that the following commute:

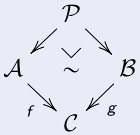


Pseudo-pullbacks

No pullbacks anymore, but pseudo-pullbacks:

Proposition

The pseudo-pullback of maps of ess exists:



Configurations of P correspond to

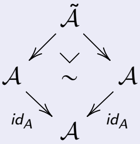
$$\mathscr{C}(A) \ni x \overset{f}{\cong} fx \overset{\theta}{\cong}_C gy \overset{g}{\cong} y \in \mathscr{C}(B)$$

that are secured

This allows us to see $\tilde{A}$ itself as an event structure with symmetry:

Proposition (Higher symmetry)

There is a canonical symmetry on $\tilde{A}$:



$$\theta \overset{\varphi,\varphi'}{\cong}_{\tilde{A}} \theta' \text{ iff}$$

$$\begin{array}{ccc} x & \overset{\theta}{\cong}_A & y \\ \overset{\varphi}{\cong}_A & & \overset{\varphi'}{\cong}_A \\ x' & \overset{\theta'}{\cong}_A & y' \end{array}$$

III. Concurrent Games with symmetry

Games with symmetry and $\sim$-pre-strategies

As in the previous part, we define

- A **concurrent game with symmetry** is an event structure with symmetry and polarities (symmetry preserves polarities)
- A $\sim$-**prestrategy** on a game $\mathcal{A}$ is a map of event structures with symmetry $\mathcal{S} \to \mathcal{A}$
- A $\sim$-prestrategy from a game $\mathcal{A}$ to $\mathcal{B}$ is a $\sim$-pre-strategy on $\mathcal{A}^{\perp} \| \mathcal{B}$.
- To update the construction of the previous section:
  - Composition: Pullbacks $\rightarrow$ pseudo-pullbacks.
  - Identity: Scott order $\rightarrow$ Scott category,

Towards a bicategory: Identity

- No natural candidate for the symmetry on $\mathbb{CC}_A$ for every $A\ldots$
- $\mathbb{CC}_A$ is too strict: it completely ignores the symmetry
- Replace the Scott order by the Scott category of configurations

$$\forall x, y \in \mathscr{C}(A), Sc(x, y) = \{\theta \in \mathscr{C}(\tilde{A}) \mid x \supseteq^{-} l\theta \overset{\theta}{\cong}_A r\theta \subseteq^{+} y\}$$

If $\theta \in Sc(x, y)$ we write $x \xrightarrow{\theta} y$.

- New **saturated copycat** $\mathbb{CC}_{\mathcal{A}}$ whose configurations are arrows from the Scott category

$$\mathscr{C}(A) \ni x \xrightarrow{\theta} y \in \mathscr{C}(A)$$

$$\mathscr{C}(\mathbb{CC}_{\mathcal{A}}) = \{(x, y, \theta) \mid x, y \in \mathscr{C}(A), y \xrightarrow{\theta} x\}$$

- Symmetry on $\mathbb{CC}_A$ is given by $\mathbb{CC}_{\tilde{\mathcal{A}}}$ (with $\tilde{A}$ considered as an event structure with symmetry)

Towards a bicategory: Composition

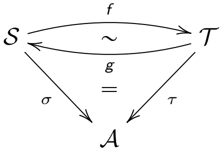
No pullbacks in event structures with symmetry, but pseudo-pullbacks! Given $\sigma : \mathcal{S} \to \mathcal{A}^{\perp} \| \mathcal{B}$ and $\tau : \mathcal{T} \to \mathcal{B}^{\perp} \| \mathcal{C}$, we form their interaction as follows:

$$
\begin{array}{ccccc}
 & & \mathcal{S} \circledast \mathcal{T} & & \\
 & \swarrow & \lrcorner & \searrow & \\
\mathcal{S} \| \mathcal{C} & & \sim & & \mathcal{A} \| \mathcal{T} \\
 & {\scriptstyle \sigma \| \mathcal{C}} \searrow & & \swarrow {\scriptstyle \mathcal{A} \| \tau} & \\
 & & \mathcal{A} \| \mathcal{B} \| \mathcal{C} & &
\end{array}
$$

Hiding yields the desired map $\tau \odot \sigma : \mathcal{S} \odot \mathcal{T} \to \mathcal{A}^{\perp} \| \mathcal{C}$.

The $\sim$-bicategory of concurrent games with symmetry

- We exploit the extra power of symmetry to have a weaker equivalence: $\sigma \simeq \tau$ iff



  with $f \circ g \sim \mathrm{id}_{\mathcal{T}}$ and $g \circ f \sim \mathrm{id}_{\mathcal{S}}$.
- What strategies behave well with respect to copycat up to that equivalence?

Theorem

A $\sim$-prestrategy $\sigma : \mathcal{S} \to \mathcal{A}$ behaves with respect to copycat iff

- *$\tilde{\sigma}$ is a strategy (in the sense of Rideau-Winskel)*
- *σ is saturated, ie. closed under the action of the symmetry of $\mathcal{A}$*

In that case, we call σ $\sim$-strategy.

- Thus we get a $\sim$-bicategory (a bicategory where coherence laws hold up to $\sim$) of games with symmetry and $\sim$-strategies.

IV. Applications

The AJM exponential

Definition

From a game with symmetry A, form $!A$ having:

- **Events**, pairs $(i, a) \in \mathbb{N} \times A$
- **Causality**,
$$(i_1, a_1) \leqslant_{!A} (i_2, a_2) \Leftrightarrow i_1 = i_2 \ \&\ a_1 \leqslant_A a_2$$
- **Consistency**,
$$\mathrm{Con}_{!A} = \bigcup_{i \in I} \{i\} \times X_i$$
- **Isomorphism family**,
$$\bigcup_{i \in I} \{i\} \times x_i \quad \overset{\theta}{\cong}_{!A} \quad \bigcup_{j \in J} \{j\} \times x_j$$
when there is a bijection $\pi : I \to J$ and isomorphisms $x_i \overset{\theta_i}{\cong}_A x_j$ with, for all $(i, a) \in \bigcup_{i \in I} \{i\} \times x_i$,
$$\theta(i, a) = (\pi(i), \theta_i(a))$$

AJM games and Classical Linear Logic

We recover (and extend) the model of [1].

Theorem

Concurrent games with symmetry form a model of classical linear logic in the sense of [1]

Proof.

We have natural maps preserving symmetry:

$$\begin{array}{llrcl}
\mu_A & : & !!A & \to & !A \\
 & & (i,(j,a)) & \mapsto & (\langle i,j\rangle, a) \\
\eta_A & : & A & \to & !A \\
 & & a & \mapsto & (0,a)
\end{array}
\qquad
\begin{array}{llrcl}
m_A & : & !A \parallel !A & \to & !A \\
 & & (1,(i,a)) & \mapsto & (2i,a) \\
 & & (2,(i,a)) & \mapsto & (2i+1,a) \\
 & & & & \\
e_A & : & 1 & \to & !A
\end{array}$$

satisfying monad/monoid laws up to symmetry. Those are lifted to $\sim$-strategies with a general construction, we get an exponential by self-duality. □

[1] P. Baillot, V. Danos, T. Ehrhard and L. Regnier, Believe it or not, AJM's games model is a model of classical linear logic, LICS'97

HO games

We also have an extension of HO games in our framework:

- an **exponential** : $⁈A$ (A an arena)
- a notion of **single-threaded** strategies on $⁈A$
- a notion of **sequential HO-innocent strategies** on $⁈A$, stable under composition

Proposition

We have a CCC $\mathcal{CHO}$ given by:

- *Objects: arenas*
- *Morphisms from A to B: correspond to negative single-threaded $\sim$-strategies on $⁈A^{\perp} \| ⁈B$*

Proposition

The sub-CCC of $\mathcal{CHO}$ consisting in deterministic and sequential HO-innocent strategies is isomorphic to the standard category of arenas and innocent strategies.

Contributions and future work

Contributions:

- Extension of the framework of Rideau-Winskel with symmetry, thus revealing interesting mathematical structure
- Extension of AJM and HO games to a concurrent setting

Future work:

- Extension with probabilities
- Connections to the metalanguage for concurrent strategies
- Applications to modeling programming languages