DM1: Transfinite philosophers

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***Exercise* 1** (The dining philosophers (Dijkstra '69))**. —** Three philosophers are sitting at a round table with a bowl of rice in the middle. For the philosophers (being a little unworldly) life consists of thinking and eating (and waiting, as we will see). To take some rice out of the bowl, a philosopher needs two chopsticks. In between two neighboring philosophers, however, there is only a single chopstick. Thus, at any time only one of two neighboring philosophers can eat. Of course, the use of the chopsticks is exclusive and eating with hands is forbidden.

Note that a deadlock scenario occurs when all philosophers possess a single chopstick. The problem is to design a protocol for the philosophers, such that the complete system is deadlock-free, i.e., at least one philosopher can eat, infinitely often. Additionally, a fair solution may be required with each philosopher being able to think and eat infinitely often. The latter characteristic is called freedom of *individual starvation*.

1. Model the scenario of three dining philosophers as a labelled transition system.
2. Can you express the following properties by LTL formulae?

   Mutual exclusion any two philosophers never eat at the same time;

   Deadlock freedom there is always at least one philosopher eating;

   No Starvation all philosophers are guaranteed to eat, sooner or later.
3. Check whether the above properties are respected by your model of the dining philosophers problem. If not, can you think of improvements?

***Exercise* 2** (My first transfinite induction)**. —** Let $(L, \leq)$ be a complete lattice. We have seen that any monotone function $f : L \to L$ has a least fixpoint and in the case that L is finite it can be computed by the lub $e_f = \vee\{f^n(\bot) \mid n \in \mathbb{N}\}$. In this exercise, all functions are assumed monotone.

1. Show that if $e_f = f(e_f)$, then e_f is the least fixpoint of f.
2. Assume L is infinite and f also satisfies that $\vee\{f(x) \mid x \in A\} = f(\vee A)$ for any countable A. Show that $f(e_f) = e_f$ In that case, we say that f is continuous.
3. Show that $(\mathbb{N}, \leq)$ is not a complete lattice. Show how to extend it with a element ω so that it becomes a complete lattice $\mathbb{N}_\infty$.[1]
4. Assume $\text{succ}(\omega) = \omega$ (only for **this** question), what is the least fixpoint of succ? Show that for any $f : \mathbb{N}_\infty \to \mathbb{N}_\infty$ we still have $f(e_f) = e_f$.
5. Show how to extend N_∞ by one element to $L_{\omega+1}$ and extend $\text{succ} : \mathbb{N} \to \mathbb{N}$ to $L_{\omega+1}$ so that $\text{succ}(e_{\text{succ}}) \neq e_{\text{succ}}$.

[1] I'm not doing my Ph.D. on $\mathbb{N}_\infty$, I promise

***Exercise* 3** (The lattice of LTL formulae)**. —** The goal of this exercise is to show that the lattice L of LTL formulae is not complete. Recall (Exercise 6 from TD2) that L is defined as the quotient of the set of LTL formulae by equivalence and the order is given by implication. We write $[\varphi]$ for the equivalence class of a formula A. We write At for the set of atomic formulae and $\Sigma = \mathscr{P}(\mathrm{At})$ for the corresponding alphabet. Given a formula φ we will write $\mathrm{Words}(\varphi) \in \mathscr{P}(\Sigma^\omega)$ for the set of words satisfying φ.

1. Show $\mathrm{Words}(\cdot)$ extends to an injective function from L to $\mathscr{P}(\Sigma^\omega)$ such that $\mathrm{Words}([\varphi]) \subseteq \mathrm{Words}([\psi])$ if and only if $[\varphi] \leq [\psi]$. Is it surjective?

2. Let p be a finite word over Σ. Show that there is a formula of LTL φ_p such that $\sigma \models \varphi_p$ if and only p is a prefix of σ. Deduce that for any finite set A of finite words over Σ, there is a formula φ_A such that $\sigma \models \varphi_A$ if and only if there is a prefix of σ in A.

3. Recall that $\phi\mathbf{U}\psi$ is the least fixpoint of $[\alpha] \mapsto [\psi \vee (\phi \wedge \mathbf{X}\alpha)]$. What is its greatest fixpoint?

4. Find a non-constant function g such that $[\mathbf{G}\varphi]$ is a fixpoint of g. Give the least and greatest fixpoint of this function.