Aperiodic safety

simon.castellan@ens-lyon.fr

***Exercise* 1** (Generalized Büchi automata)**. —** In this exercise we fix $\mathrm{At} = \{a, b\}$.

1. Draw carefully the generalized Büchi-automaton related to the formula $a\mathbf{U}(\mathbf{F}b)$. Describe the states, transitions and acceptance sets of it.

2. For the connector $\mathbf{F}$ explain how one could extend the construction of a GNBA to this connector without resorting to the encoding through $\mathbf{U}$. Extend the construction of the automaton $\mathcal{G}_\varphi$ corresponding to a formula φ and its correctness proof.

3. Use this new construction to build a GNBA asscoiated to the formula $\mathbf{G}(a \to \mathbf{F}b)$ (using the encoding $\mathbf{G}$ in terms of $\mathbf{F}$).

***Exercise* 2** (Aperiodicity)**. —** Recall that a finite monoid is a finite set M equipped with a binary operation (called *multiplication*) $\cdot : M^2 \to M$ which is associative: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in M$ and a neutral element $e \in M$ with $x \cdot e = e \cdot x = x$ for all $x \in M$.

A finite monoid M is *aperiodic* when there exists $n \in \mathbb{N}$ such that for all $x \in M$ such that $x^n = x^{n+1}$.

1. Show that if M and M' are aperiodic monoids then $M \times M'$ is aperiodic (with the pointwise multiplication)

Let M be a finite monoid and $h : \Sigma^* \to M$ be a homomorphism. Given $u, v \in \Sigma^\infty$ (ie. finite or infinite words) we say that $u \simeq_h v$ when $u = u_0 \dots u_n \dots$ and $v = v_0 \dots v_n \dots$ with $h(u_i) = h(v_i)$ for every i, and each of the u_i and v_i are finite word over Σ. The sequences are always infinite but can be ultimely equal to ϵ to handle the case where u and v are finite. Note that if u and v are finite this amounts to $h(u) = h(v)$ because h is a homomorphism.

A language $L \subseteq \Sigma^\infty$ is h-saturated when $u \in L$ and $u \simeq_h v$ implies $v \in L$.

A language is aperiodic when it is h-saturated (or h saturates the language) for $h : \Sigma^* \to M$ with M a finite aperiodic monoid. (Note in the literature it is said that h
recognizes L).

2. Show that for all $a \in \Sigma$, $\{a^\omega\}$ is an aperiodic language, as well as $\emptyset$.

3. Show that if L is h-saturated for $h : \Sigma^* \to M$ and L' is h'-saturated for $h' : \Sigma^* \to M'$ then L and L' are **both** h''-saturated for a third $h'' : \Sigma^* \to M''$.

4. Show that if L and L' are aperiodic languages, then so are $L \cap L'$, $L \cup L'$, $\Sigma^\infty \setminus L$.

5. Let M and N be aperiodic monoids along with $h : \Sigma^* \to M$ and $h' : \Sigma^* \to M'$. We write $\diamond(h, h')$ for the map
$$\Sigma^* \to \mathscr{P}(M \times N), w \mapsto \{(h(u), h'(v)) \mid u \cdot v = w\}$$
Write $M_{h,h'}$ for the image of the map $\diamond(h, h')$.

(a) Define a monoid structure on $M_{h,h'}$ such that $\diamond(h,h')$ is an homomorphism.

(b) Show $M_{h,h'}$ is an aperiodic monoid.

(c) Show that if $\sigma \simeq_{\diamond(h,h')} \sigma'$ then we have $\sigma^n \simeq_{\diamond(h,h')} \sigma'^n$ for every $n \in \mathbb{N}$ and *infinite* words $\sigma, \sigma' \in \Sigma^\omega$.

(d) Assume that for LTL-formulae φ and ψ, their associated languages is h-saturated for $h : \Sigma^* \to M$. Show that

$$\diamond(\diamond(h,h),h) : \Sigma^* \to M_{\diamond(h,h),h}$$

saturates the language of $\varphi \mathbf{U} \psi$

6. Show that any language recognized by a formula of LTL is aperiodic.

We admit the converse (any aperiodic language can be recognized by a LTL formula).

***Exercise* 3** (Topological safety)**.** — Recall that a topology on a set X is a set $\mathcal{T} \subseteq \mathscr{P}(X)$ of open sets of X, containing $\emptyset$, X and stable by arbitrary unions and finite intersections. A closed set is the complement of an open set, and a dense set is a set that has a non-empty intersection with every non-empty open set.

1. Show that every subset of a topological space $(X, \mathcal{T})$ is an intersection of a closed set and a dense set.

2. Write the definition of the topology induced by the metric seen during the lectures. What are its closed sets?

3. Deduce that every linear-time property is the intersection of a liveness property and a safety property.

4. Recall that the closure of a set X in a topology $\mathcal{T}$ is the smallest closed set containg X. (Formally it is the intersection of all closed sets containg X). Show that the closure of an aperiodic language is aperiodic using the construction of Question 2.6.

5. From the characterisation of LTL formulae of Exercise 2, and Question 4, deduce that any LTL-formula φ is equivalent to a formula $\varphi_s \wedge \varphi_l$ where Words(φ_s) is a safety property and Words(φ_l) is a liveness property. (Hint: Use a trick similar to question 1).