TD1: Reminder on orders, and transition systems

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1 Reminder on orders

***Exercise* 1** (Orders and upper bounds)**. —** Let $(L, \leq)$ be an order. We recall the following definitions:

- An element $x \in L$ is an *upper bound* of a subset $A \subseteq L$ whenever $x \geq y$ for all $y \in A$
- An element $x \in L$ is the *least upper bound* (lub) of a subset $A \subseteq x$ whenever $x \leq y$ for all upper bound y of A

Dually, we define a *lower bound* of A to be an upper bound of A in $(L, \geq)$ and a *greatest lower bound* (glb) as a least upper bound of A in $(L, \geq)$.

Recall that the least upper bound is unique if it exists (and therefore so are greatest lower bounds).

1. Show that if a subset of $(\mathbb{N}, \leq)$ has an upper bound, it has a least upper bound. Does the same property hold in $(\mathbb{R}, \leq)$?
2. Show that for any $a, b \in \mathbb{N}$, $\{a, b\}$ both have a greatest lower bound and a least upper bound in $(\mathbb{N}, |)$.

A function $f : (L, \leq_L) \to (L', \leq_{L'})$ is monotone whenever $f(x) \leq_{L'} f(x')$ for every $x \leq_L x'$. Let $f : (L, \leq_L) \to (L', \leq_{L'})$ be a monotone function.

3. Let x be an upper bound of $A \subseteq L$. Show $f(x)$ is an upper bound of $f(A) = \{f(x) \mid x \in A\}$. Is the same true for least upper bound?
4. We say that two orders $(L, \leq_L)$ and $(L', \leq_{L'})$ are *isomorphic* whenever there exists two monotone maps $f : L \to L'$ and $g : L' \to L$ such that $f \circ g = \mathrm{id}_{L'}$ and $g \circ f = \mathrm{id}_L$.

   Is it true that $(L, \leq_L)$ and $(L', \leq_{L'})$ are isomorphic whenever there exists a monotone bijection $f : L \to L'$?

***Exercise* 2** (Lattices, complete lattices)**. —** We say an order $(L, \leq)$ is a lattice whenever every finite non-empty subset of L has both a least upper bound $\vee L$ and a greatest lower bound $\wedge L$.

A lattice is complete if this property holds for any subset, not necessarily finite. In particular, applying this to the empty set, it follows that L has a minimum element (written $\bot$) and a maximum element ($\top$).

1. Show that L is a lattice if and only if there exists binary operators $\vee : L^2 \to L$ and $\wedge : L^2 \to L$ satisfying some equations to be spelled out. (Hint: show your equations entail $a \wedge b = a$ iff $a \vee b = b$)
2. Show that every total order is a lattice.
3. Is $(\mathbb{N}, \leq)$ a complete lattice? Is $(\mathbb{R} \cup \{-\infty, +\infty\}, \leq)$ a complete lattice?
4. For any set X, show $\mathscr{P}(X)$ is a complete lattice.

***Exercise* 3** (Knaster-Tarski theorem: Fixpoint of monotone functions in complete lattices)**.** — Let $(L, \leq)$ be a complete lattice and $f : L \to L$ a monotone map.

1. Show $\wedge\{x \in L \mid f(x) \leq \times\}$ is the least fixpoint of f. Show f has a greatest fixpoint.
2. Assume L is finite . Show the least fixpoint is equal to $\vee\{f^n(\perp) | n \in \mathbb{N}\}$.
3. (*hard*) Is it still the case when L is not finite?

2 Transition systems

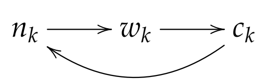
***Exercise* 4** (Modelisation of systems and linear properties)**.** — Consider the following transition systems:

1. The problem of the ferryman. A ferryman must help a goat, a cabbage and a wolf cross a river. His ferry is only big enough to carry one passenger.

   (a) Give a transition system modelling this system.

   (b) We now add the constraint that whenever either the goat and the cabbage or the goat and the wolf are on the same side of the river, the ferryman must be on this side too. Give a linear-time property to characterize the paths satisfying this constraint. What form does the property have?

2. Consider the family of transition systems (P_k) of the form



   P_k is a rough approximation of a program performing some calculations (n_k), waits to enter a critical section (w_k) and then enters it (c_k), and loops.

   (a) Draw the transition system corresponding to the interleaving $P_1 \parallel P_2$.

   (b) Notice how in this transition system, the state (c_1, c_2) is reachable. Write the LP property corresponding to this property of mutual exclusion which is not satisfied here: no two processes are in the critical section at the same time.

   (c) By removing some transitions from $P_1 \parallel P_2$, form a new transition system P that satisfies this property.

   (d) We would like to ensure a liveness property: if both processes are waiting, then at least one of them is allowed in the critical section. Capture this behavior with a linear-time property. Does it have the same shape as the previous ones?

   (e) Another class of interesting properties are *fairness properties*. Define a linear time property corresponding to the following "if one of the two process if waiting infinitely often, it enters the critical section infinitely often". Is it satisfied by P ? by $P_1 \parallel P_2$?