# TD11 – Galois connection

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A Galois connection between two orders $A$ and $B$ is a pair of monotone maps ($\alpha : A \to B, \beta : B \to A$) with $\alpha(a) \leq_B b$ if and only if $a \leq_A \gamma(b)$; in this case we write $\alpha \dashv \gamma : A \to B$.

***Exercise* 1** (Warmup). —

1. Let $(\alpha : A \to B, \gamma : B \to A)$ be a pair of monotone maps such that $\gamma \circ \alpha(c) \geq c$ and $\alpha \circ \gamma(c) \leq c$. Show that $\alpha \dashv \gamma$.
2. Let $f : \mathscr{P}(X) \to Y$ be a monotone map such that $f \dashv g$ for some $g$. Give an explicit formula for $g$ in terms of $f$.
3. Let $f : X \to Y$ be a function between sets. There is a map $f_! : \mathscr{P}(Y) \to \mathscr{P}(X)$ defined by $A \mapsto f^{-1}A$. Is there a map $g : \mathscr{P}(X) \to \mathscr{P}(Y)$ such that $g \dashv f_!$? such that $f_! \dashv g$?

***Exercise* 2** (Examples of abstract domains). — For the concrete domain here we use the order $\mathscr{P}(\mathbb{Z})$.

For the two following properties, explain how to build an order $A_0$ and a Galois connection $\alpha \dashv \gamma : \mathscr{P}(\mathbb{Z}) \to A_0$ that captures the desired information, and give the addition operator on this order (as a monotone function $A_0 \times A_0 \to A_0$):

- Whether the value is positive, strictly positive, nogative or strictly negative
- A possible interval in which the value is contained.

***Exercise* 3** (Back to Imp). — Remember that one can interpet Imp inside the $\omega$-CPO of partial functions from $\mathscr{P}(\mathscr{V} \to \mathbb{Z})$ to itself. This is the concrete semantics.

1. Write an Imp program that computes in the variable $o$ the lgcd of the variables $x$ and $y$.
2. Compute its concrete semantics.
3. Explain how to turn the abstract domain of intervals into an abstraction for Imp programs.
4. Compute the interpretation of the previous program in this model.

***Exercise* 4** (From Closure Operators to Galois Connections). — Let $(P, \leqslant)$ be a complete lattice and let $c : P \to P$ be a *closure operator* on $P$, ie a monotone idempotent function satisfying $x \leq c(x)$ for $x \in P$.

1. Define the set of *closed elements* (notation $P_c$) of $P$.
2. Prove that, for all $x \in P$, $c(x) = \bigwedge_P \{y \in P_c \mid x \leqslant y\}$.
3. Prove that $P_c$ is a complete lattice.

   Hint: prove that $\bigwedge_{P_c} S$ is given by $\bigwedge_P S$ and $\bigvee_{P_c} S$ by $c(\bigvee_P S)$ for $S \subseteq P_c$
4. Given $P$ and $c$, define a Galois Connection having $P$ as concrete domain.