TD12: This is the end

simon.castellan@ens-lyon.fr

We recall that $\mathscr{M}$ is the set of functions from the set of variables to $\mathbb{N}$.

***Exercise* 1** (Warmup). — 1. Given a Imp command C and a boolean expression b, recall the denotational semantics of the Imp term `while` b `do` C `done` in the ω-CPO $[\mathscr{M}_\perp \to \mathscr{M}_\perp]$.

2. Use this formula to compute the denotational semantics of

$r := n;$`while` n `> 1 do n := n - 1; r := n * r done`

***Exercise* 2** (More on abstract interpretation). — Recall from the lectures the abstract domain $A_0 = \{\perp, \top, \texttt{odd}, \texttt{even}\}$ and the Galois connection $\alpha \dashv \gamma : \mathscr{P}(\mathbb{Z}) \to A_0$.

1. Recall the relation R_α from $\mathscr{P}(\mathbb{Z})$ to A_0 defined by $aR_\alpha c$ when $a \leq \gamma c$. Show that any function $f : \mathscr{P}\mathbb{Z} \to \mathscr{P}\mathbb{Z}$ can be lifted to a function $\bar{f} : A_0 \to A_0$ such that $f(R_\alpha \to R_\alpha)\bar{f}$.

2. By induction on the arithmetir expression, define the abstract interpretation of a as a function from $\mathscr{V} \to A_0$ to itself.

3. Compute the lift of $f := n \mapsto \lceil n/2 \rceil$ and $g := n \mapsto \lfloor n/2 \rfloor$. Deduce the abstract interpretation of $a := \lceil x/2 \rceil; b. = \lfloor x/2 \rfloor; x := a + b$.

4. Compute the abstract interpteation of $a := \lceil x/2 \rceil; b. = \lfloor x/2 \rfloor; x := \lceil x/2 \rceil + \lfloor x/2 \rfloor$. Can you deduce a (non-)property of $\bar{\cdot}$?

5. Consider the same program as in question 1.3:

   $r := n;$`while` n `> 1 do n := n - 1; r := n * r done`

   Compute its abstract interpretation.

6. How could we change the program (not changing the final version of r) to get a non-trivial abstact semantics?

***Exercise* 3** (Galois connections). — 1. Let C and A be orders with a Galois connection $\alpha \dashv \gamma : C \to A$. Show that if $c \in C$, then $\alpha(c)$ is a fixpoint of $\alpha \circ \gamma$ and if $a \in A$ then $\gamma(a)$ is a fixpoint of $\gamma \circ \alpha$. (This result is valid for any Galois connection.)

2. Let C and A be lattices and $\alpha \dashv \gamma : C \to A$ be a Galois connection. Show that $\alpha(\perp) = \perp$ and $\alpha(c \vee c') = \alpha c \vee \alpha c'$. What can we say about γ?

3. Let Ω be a set and A a subset of Ω. Consider $f : \mathscr{P}(\Omega) \to \mathscr{P}(\Omega), X \mapsto X \cap A$. Does it have a right adjoint? A left adjoint?

4. Let $(P, \leqslant)$ be a complete lattice and let $c : P \to P$ be a *closure operator* on P, ie a monotone idempotent function satisfying $x \leq c(x)$ for $x \in P$. Show that the set of fixed points of c is a complete lattice.

   Define a Galois connection whose concrete domain is P.

5. What is the abstract domain corresponding to the closure operator on $\mathscr{P}\mathbb{Z}$ mapping a subset of integers to its convex hull?