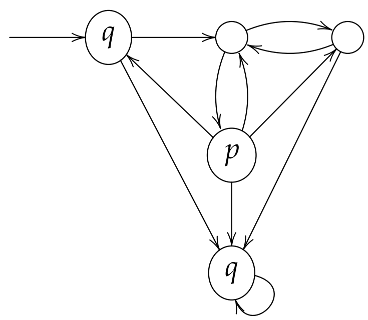
TD2: Playing with LTL

simon.castellan@ens-lyon.fr

***Exercise* 1** (Warmup). — Consider the following transition system $\mathcal{T}$ (At $= \{p, q\}$):



$$\begin{aligned}
\varphi_1 &= \mathbf{G}q \\
\varphi_2 &= \mathbf{F}(\neg p) \\
\varphi_3 &= p\mathbf{U}q \\
\varphi_4 &= q\mathbf{U}\mathbf{X}(p \wedge \neg q) \\
\varphi_5 &= \mathbf{X}\neg q \wedge \mathbf{G}(\neg p \vee \neg q)
\end{aligned}$$

For φ_i, with $i \in \{1, 2, 3, 4, 5\}$:

1. find a path from the initial state which satisfies φ_i;
2. determine whether $\mathcal{T} \models \varphi_i$.

***Exercise* 2** (LTL Expressivity). — Let p and q be some formula. Show how to express using LTL the following LT property:

- (Safety) p and q never occur at the same time
- (Liveness 1) the formula p is true infinitely often (on this path)
- (Liveness 2) if p occurs infinitely often, then so does r
- (Liveness 3) any time p holds, q will hold at some point in the future

***Exercise* 3** (Equivalence in LTL). — Prove or give a counterexample (ie. a transition system) to the following formulas:

$$\mathbf{G}(\varphi \wedge \psi) \equiv \mathbf{G}\varphi \wedge \mathbf{G}\psi$$
$$\mathbf{F}(\varphi \wedge \psi) \equiv \mathbf{F}\varphi \wedge \mathbf{F}\psi$$
$$\mathbf{G}(\varphi \vee \psi) \equiv \mathbf{G}\varphi \vee \mathbf{G}\psi$$
$$\mathbf{FF}\varphi \equiv \mathbf{F}\varphi$$
$$\mathbf{FG}\varphi \equiv \mathbf{GF}\varphi$$
$$\mathbf{GFGF}\varphi \equiv \mathbf{GF}\varphi$$
$$\mathbf{XG}\varphi \equiv \mathbf{GX}\varphi$$
$$\mathbf{XF}\varphi \equiv \mathbf{FX}\varphi$$

***Exercise* 4** (Expressivity of LTL). —

1. Express $\mathbf{F}\varphi$ in terms of the until operator.
2. Express the until operator in terms of other temporal modalities?
3. Prove that there exists a LTL formula φ such that $\mathbf{X}\varphi$ is not equivalant to a formula of the form $\mathbf{F}\psi$ and $\mathbf{G}\psi$ for any LTL formula ψ.
4. Find all the minimal sets of adequate modalities (a set of modality is adequate when it can express all the other conncectives).

***Exercise* 5** (Intuitionistic logic is back). — Recall that a state s of a LTS $\mathcal{T}$ satisfies φ ($s \models \varphi$) whenever every path starting at s satisfies φ.

1. Show that for any path σ, $\sigma \models \varphi$ or $\sigma \models \neg\varphi$. Deduce that every model $\mathcal{M}$ satisfies $\varphi \vee \neg\varphi$
2. Given a LTS $\mathcal{T}$ and a state $s \in \mathcal{T}$, do we have that $s \models \varphi$ or $s \models \neg\varphi$. Deduce that we don't have that $\mathcal{T} \models \varphi$ or $\mathcal{T} \models \varphi$.
3. Show that $\models \neg F(\varphi) \Rightarrow \mathbf{G}(\neg\varphi)$. Do we have that if $\mathcal{T}$ does not satisfy $\mathbf{F}\varphi$, it satisfies $\mathbf{G}(\neg\varphi)$.

***Exercise* 6** (The lattice of LTL formulae). — Write L for the quotient of the set of LTL formulae by the equivalence relation $\equiv$. Given a formula φ we write $[\varphi]$ for its equivalence class in L.

1. Show that implication defines an pre-order on the set of formulas that induces an order on L that we write $\leq$.
2. Show $(L, \leq)$ is a lattice with a maximum and minimum element.
3. Show $\varphi\mathbf{U}\psi \equiv \psi \vee (\varphi \wedge \mathbf{X}(\varphi\mathbf{U}\psi))$. Deduce a monotone non constant function $f : L \to L$ such that $[\varphi\mathbf{U}\psi]$ is a fixpoint of f.
4. Is it the least fixpoint of f?
5. Find a non constant function g such that $[\mathbf{G}\varphi]$ is a fixpoint of f. Is it a least fixpoint?
6. (*hard*) Is L a complete lattice?