TD3: ω-languages

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ω-regular expression An ω-regular expression is an expression of the form $e = e_1 \cdot f_1^\omega + \ldots + e_n \cdot f_n^\omega$. The language it denotes is $\mathcal{L}(e) = \bigcup_{i=1}^n \ell(e_i) \cdot \ell(f_i)^\omega$.

***Exercise* 1** (Warmup). — Give ω-regular expressions corresponding to the following ω-languages over $\Sigma = \{a, b\}$:

1. The language of infinite words containing a finite number of b
2. The language of infinite words containing an infinite number of a

Give ω-regular expressions corresponding to the following linear-time properties (alphabet = $\mathscr{P}(\{a, b\})$)

1. a is always satisfied and b is never satisfied
2. Eventually, a is true
3. a will always be true from a certain point on

***Exercise* 2** (Regular safety properties). — Let $\mathcal{S}$ and $\mathcal{T}$ be transition systems and Φ a safety property whose bad prefix language is written Π_Φ.

1. Show that $\mathcal{T} \models \Phi$ if and only if $\text{FTraces}(T) \cap \Pi_\Phi = \emptyset$
2. Show that $\text{FTraces}(T) \subseteq \text{FTraces}(S)$ if and only if, for every safety property Φ, $\mathcal{S} \models \Phi \Rightarrow \mathcal{T} \models \Phi$.

***Exercise* 3** (Characterisation of ω-rational language). — Show that the class of ω-rational languages is exactly the class of ω-languages described by ω-regular expressions.

***Exercise* 4** (On Büchi-automata). — Let $\mathcal{A} = (Q, I, \delta, F)$ be a finite (non-deterministic) automaton. We say that $\mathcal{A}$ Büchi-accepts an infinite word $\sigma \in \Sigma^\omega$ whenever there is an infinite sequence $(q_n)_{n\in\mathbb{N}}$ of states of $\mathcal{A}$ with

1. $q_0 \in I$
2. $(q_i, \sigma_i, q_{i+1}) \in \delta$
3. The set $\{q_n \in F \mid n \in \mathbb{N}\}$ is infinite.

The ω-language associated to $\mathcal{A}$ is the set of infinite words it Büchi-accepts. It will be denoted $\mathcal{L}(\mathcal{A})$ (and $\ell(\mathcal{A})$ for the language of finite words it recognizes).

1. Give automata whose ω-language correspond to the languages of Exercise 1.
2. Show that every ω-rational language is recognized by an automaton.
3. Conversely, using the characterization of ω-rational language of Exercise 3, show that the ω-language recognized by an non-deterministic automaton $\mathcal{A}$ is ω-rational.