TD7: Orders and topology

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Let $(X, \mathcal{T})$ be a topological space. We recall the definitions:

- X is T_0 whenever for all distinct $x, y \in X$ there exists an open $U \in \mathcal{T}$ such that $U \cap \{x, y\}$ is a singleton.
- A neighborhood of $x \in X$ is an open set containg x
- A closed neighborhood of $x \in X$ is a closed set containing a neighborhood of x.
- X is T_1 whenever for all distinct $x, y \in X$ there exists open sets $U, V \in \mathcal{T}$ such that $U \cap \{x, y\} = x$ and $V \cap \{x, y\} = \{y\}$.
- X is T_2 when for all distinct $x, y \in X$ there exists disjoint $U, V \in \mathcal{T}$ such that $x \in U, y \in V$.
- $\mathcal{T}$ induces a preorder (called *specialization preorder*) on X as follows $x \sqsubseteq_{\mathcal{T}} y$ when for all open $U \in \mathcal{T}$, the presence of x in U implies that of y ($x \in U \Rightarrow y \in U$)

***Exercise* 1** (Warmup)**. —**

1. Show that $\mathbb{R}$ with the usual topology is T_2. What is the preorder $\sqsubseteq_{\mathbb{R}}$?
2. Set $X = \mathbb{N}$. Consider $\mathcal{T}$ to be the set of cofinite subsets of $\mathbb{N}$ (ie. those whose complement is finite). Show $(\mathbb{N}, \mathcal{T})$ is a topology which is T_1 but not T_2. What is the preorder $\sqsubseteq_{\mathcal{T}}$?
3. Show that $\mathcal{T} = \{\emptyset, \{0\}, \{0, 1\}\}$ is a topology on $\{0, 1\}$ called the Sierpi ń ski space. Show it is T_0 but not T_1. What is the preorder $\sqsubseteq_{\mathcal{T}}$?
4. The topology on Σ^ω comprising the sets of the form $A.\Sigma^\omega$ (for A a language of finite word). Is it T_0, T_1, T_2? What is its specialization preorder?

***Exercise* 2** (Hierarchy of conditions)**. —** We fix a topological space $(X, \mathcal{T})$.

1. Show that the following are equivalent
   - (a) $(X, \mathcal{T})$ is T_2
   - (b) For all x, the intersection of al the **closed** neighborhood is the singleton $\{x\}$
   - (c) The diagonal (ie. the set $\Delta = \{(x, x) \in X\}$) is closed inside the product space $X\times$
2. Show the following are equivalent:
   - (a) $(X, \mathcal{T})$ is T_1
   - (b) For all $\{x\}$, the intersection of al the neighborhood is the singleton $\{x\}$
   - (c) For all $x \in X$, $\{x\}$ is closed.
3. Show the following are equivalent:
   - (a) $(X, \mathcal{T})$ is T_0

(b) $\sqsubseteq_{\mathcal{T}}$ is a partial order.

4. Deduce that $T_2 \Rightarrow T_1 \Rightarrow T_0$
5. Show that if $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ is continuous then it is monotonic with respect to the specialization preorder.

***Exercise* 3** (From orders to topology)**.** — Let $(X, \leq)$ be a partial order.

1. Construct a topology $\mathcal{T}_\alpha$ on X whose open sets are the subsets of X upward closed $(x \in U \& x \leq y \Rightarrow y \in U)$, Show its specialization order is $\leq$.
2. Show that a monotone function $f : (X, \leq) \to (Y, \leq)$ is continuous for the induced topologies on X and Y via Question 1.
3. Show that the set $\mathcal{T}_\omega$ comprising for each $A \subset \mathscr{P}_f(X)$ (a set of finite subsets of X) the open set

$$\mathcal{O}_A = \bigcup_{Z \in A} X \setminus (\downarrow Z)$$

is a topology on X with specialization order $\leq$.

Does it satisfy the property of Question 2?

4. Is the identity function a continuous function between $(X, \mathcal{T}_\alpha)$ and $(X, \mathcal{T}_\omega)$?