TD8: Orders and topology

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We recall the syntax of the language IMPwhich is a minimal imperative language. We assumed fixed a (countably infinite) set of variable names $\mathscr{V}$ and use r, s, t to range over them. Its syntax is as followed

$a, a' ::=$	(*Arithmetic expressions*)
$\mid \underline{n}$	Integer litterals
$\mid a + a' \mid a \times a' \mid a - a'$	Addition, Soustraction and Multiplication
$\mid r$	Variable dereferenciation
$b, b' ::=$	(*Boolean expressions*)
$\mid a = a'$	Integer equality
$\mid b \wedge b' \mid b \vee b' \mid \neg b$	Conjunction, Disjunction and Negation
$C, C' ::=$	(IMP*commands*)
$\mid r := a$	Affectation
$\mid \texttt{if}\ b\ \texttt{then}\ C\ \texttt{else}\ C'$	Conditionals
$\mid \texttt{while}\ b\ \texttt{do}\ C\ \texttt{done}$	Unbounded loops
$\mid \texttt{skip}$	No-op
$\mid C; C'$	Sequencing
$(\mid \texttt{while}_n\ b\ \texttt{do}\ C\ \texttt{done}$	Bounded loops)

A pure IMP-term is a term which does not feature the last two constructions. A bounded IMP-term is a term of Imp which does not feature unbounded while loops (but is allowed to contain bounded loops). Write IMP for the set of IMP commands.

We call a memory state a map $\mathscr{V} \to \mathbb{N}$ assigning to each variable a value.

***Exercise* 1** (Operational semantics). —

1. Define the natural semantics (big-step semantics) of this language (of the form $\langle C, \sigma \rangle \Downarrow \sigma'$ with σ, σ' two memory states)
2. Define the small-step semantics $\langle C, \sigma \rangle \to \langle r, \sigma' \rangle$ with $r \in \text{IMP} \cup \{\bullet\}$ where $\bullet$ means termination.
3. If C is an IMP command, write C_n for the command C where every unbounded while loops have been replaced by while$_n$. Show that if $\langle C, \sigma \rangle \Downarrow \sigma'$ then there exists $n \in \mathbb{N}$ such that $\langle C_n, \sigma \rangle \Downarrow \sigma'$.
4. Show that $\sigma, c \Downarrow \sigma'$ if and only if $\sigma, c \to^* \sigma'$.
5. (Determinism) Show that if $\sigma, c \Downarrow \sigma'$ and $\sigma, c \Downarrow \sigma''$ then $\sigma' = \sigma''$.
6. Define a relation on commands: $c \prec c'$ whenever for all σ, σ' and $\sigma, c \Downarrow \sigma'$, then $\sigma, c' \Downarrow \sigma'$. Is $\prec$ an order?
7. Describe the quotient of the IMP commands by the equivalence generated by $\prec$. What is the order induced on it?

8. Show that the sequencing operation ($(C, C') \mapsto C; C'$) on IMP commands induces an operation on the quotient. What does it correspond to?

9. Let $C \in$ IMP. Show that in the quotient set, that the least upper bound for the induced order of the set $\{C_n \mid n \in \mathbb{N}\}$ is C.

***Exercise* 2** (Failure of IMP). — We wish to extend IMP with an instruction abort that terminates the program being run (and thus throwing away the rest of the calculation).

Propose ways to extend the natural semantics and the small-step semantics to account of this behaviour.

***Exercise* 3** (From orders to topology). — Let $(X, \leq)$ be a partial order.

1. Construct a topology $\mathcal{T}_\alpha$ on X whose open sets are the subsets of X upward closed $(x \in U \& x \leq y \Rightarrow y \in U)$, Show its specialization order is $\leq$.

2. Show that a monotone function $f : (X, \leq) \to (Y, \leq)$ is continuous for the induced topologies on X and Y via Question 1.

3. Show that the set $\mathcal{T}_\omega$ comprising for each $A \subset \mathscr{P}_f(X)$ (a set of finite subsets of X) the open set

$$\mathcal{O}_A = \bigcup_{Z \in A} X \setminus (\downarrow Z)$$

   is a topology on X with specialization order $\leq$.

   Does it satisfy the property of Question 2?

4. Is the identity function a continuous function between $(X, \mathcal{T}_\alpha)$ and $(X, \mathcal{T}_\omega)$?