TD9: ω-CPOs

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We recall that a ω-cpo is a (partial) order $(A, \leq)$ such that every increasing sequence $x_1 \leq \ldots \leq x_n \leq \ldots$ has a least upper bound that we write $\sup(x_n)_n$ and A has a least element written $\perp_A$. A Scott-continuous function f between ω-CPOS $(A, \leq)$ and $(B, \leq)$ is a monotone function such that for every increasing sequence we have:

$$f(\sup(x_n)_n) = \sup(f(x_n))_n$$

This makes sense because $(f(x_n))$ is an increasing sequence of B.

***Exercise* 1** (Warmup)**. —**

1. Let X be a set. Show that $\{\perp\} \cup X$ equiped with the order generated by $\perp \leq x$ is a ω-CPO written $X_\perp$.
2. Let D, D' be ω-CPOs. Show that the set of Scott-continuous functions from D to D' (with an order to be determined) is an ω-CPO $[D \to D']$.
3. Let X, Y be sets. Show that the ω-CPO $[(X, =) \to Y_\perp]$ correspond to partial functions from X to Y ordered by graph inclusion.
4. Show that eval : $[A \to B] \times A \to B$ is a continuous function.

***Exercise* 2** (Scott topology)**. —** Let $(X, \leq)$ be an ω-CPO.

1. Construct a topology $\mathcal{T}_\alpha$ on X whose open sets are the subsets of X upward closed ($x \in U \& x \leq y \Rightarrow y \in U$), Show its specialization order is $\leq$.
2. Show that a monotone function $f : (X, \leq) \to (Y, \leq)$ is continuous for the induced topologies on X and Y via Question 1.
3. Show that the set of subsets $\mathscr{O}$ of X satisfying
   (a) $\mathscr{O}$ is upward closed ($x \leq y$ and $x \in \mathscr{O}$ implies $y \in \mathscr{O}$)
   (b) If $\sup(x_n)_n \in \mathscr{O}$ then there exists $n \in \mathbb{N}$ such that $x_n \in \mathscr{O}$

   forms a topology on X whose specialization preorder is $\leq$ called **the Scott topology**.
4. Compare the topologies of Question 1 and the Scott topology.
5. Let $x \in X$. Show that $\{a \in X \mid \neg(a \leq x)\}$ is an open of the Scott topology on X
6. Let $f : X \to Y$ be a functions between two ω-CPOs. Show that f is Scott-continuous if and only if f is continuous for the Scott topologies on X and Y.

***Exercise* 3** (The Scott model of the pure λ-calculus)**. —** Let $(S, \leq)$ be an order on a countable set S. We consider $I(S)$ the set of down-closed subsets of S.

1. Show $(I(S), \subseteq)$ is a ω-CPO.

2. Let $f : I(S) \to I(S)$ be a Scott-continuous function. We associate to f the set $tr(f)$ defined by

$$tr(f) = \{(u, b) \mid b \in f(\downarrow u) \& u \in \mathscr{P}_f(S)\}$$

where $\downarrow u$ is the downclosure of u.

Show the injectivity of the mapping $f \mapsto tr(f)$.

3. Can you find an ω-CPO D such that there exists a continuous injection $[D \Rightarrow D] \to D$ and D has strictly more that one element?