TD 11 : The midsequent's theorem and Herbrand's theorem

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We set a first-order $\mathscr{L}$ containg term constructors $g_1, \ldots, g_n$ and predicates $P_1, \ldots P_k$. To simplify the explanations, we assume that there is a constant $*$ in the language.

Exercice 1 *One-sided sequent calculus*

In this exercice, we introduce the one-sided sequent calculus that aims at giving a minimal presentation of classical logic. By iterating the negation rule, from a derivation of $\Gamma \vdash \Delta$, one can build a derivation of $\vdash \neg\Gamma, \Delta$. The converse can also be shown: if $\vdash \neg\Gamma, \Delta$ is derivable, then so is $\vdash \Gamma, \Delta$.

This leads us to define, what we call the *one-sided* sequent calculus that deal with sequent of the form $\vdash \Gamma$. The main advantage of this system is that it takes advantage of the symmetry of connectors to get a very minimal system where each connector has one rule, the introduction rule. Because there is no left sequent, the rules for negation do not apply anymore and negation is *defined* by de Morgan's Laws. The destruction rule (or left introduction in the sequent calculus) of a connector is specified by giving it a *dual connector* through the associated de Morgan's Law. From now on, we consider the following class of formulas:

$$A, B ::= P(x_1, \ldots, x_n) \mid \neg P(x_1, \ldots, x_n) \mid A \wedge B \mid A \vee B \mid \forall x A \mid \exists x A$$

We use $A \Rightarrow B$ as an abbreviation for $\neg A \vee B$.

1. By induction on the structure of formulas, define the negation $\neg A$ of a formula A.
2. Define the rules for the one-sided sequent calculus to prove judgement of the form $\vdash \Gamma$. (How to write the axiom rule?)
3. Show that $\Gamma \vdash \Delta$ in the usual sequent calculus iff $\vdash \neg\Gamma, \Delta$ is derivable in your new system.

In the following we will use the calculus defined at the question 1.3.

Exercice 2 *Quantifiers and commutations*

The active formula of a rule is the formula introduced by the rule.

1. A formula is in prenex form iff it is of the form $\overrightarrow{Qx}A$ where $\overrightarrow{Qx}$ is a serie of quantifiers and A is a quantifier free formula. Show that every formula is provably equivalent to a prenex formula.
2. Show that two applications of two different rules with a different active formula commute. (Meaning: the premisse of the first rule are not active in the second one)

Exercice 3 *Midsequent's theorem*

We wish now to prove the midsequent's theorem.

Theorem 1 ***Midsequent's theorem.*** — *If $\vdash \Gamma$ is derivable and only contains prenex formulas, then there exists a cut-free derivation $\pi :\vdash \Delta$ whose structure (from the conclusion to the axiom leaves) is:*

- *Structural and quantifier rules until we reach a proof π' of a sequent $\vdash \Delta'$ where Δ' is quantifier-free*
- *Logical rules (axioms, and, or)*

The sequent $\vdash \Delta'$ is called a midsequent for $\vdash \Gamma$.

1. Why do we need the prenex hypothesis?
2. Consider the special case $B(x, y) = D(x) \Rightarrow D(y)$ where D intuitively means "x drinks". Then $\exists x \forall y B(x, y)$ is the drinker paradox (there exists someone such that if he is drinking, then every one is).

   Give a proof $\vdash \exists x \forall y B(x, y)$ that satisfies the condition of the midsequent theorem (structural rules, then quantifier rules, then logical rules).
3. By virtue of question 2.1, we consider derivations of $\vdash \Gamma$ that is cut-free, has atomic axioms and without quantifier weakening. We consider the following measure on such derivations $\mu(\pi \vdash \Delta)$ which the sum of the number of rules $\wedge$ and $\vee$ for all quantifiers rules appearing in π. In particular, if $\mu(\pi) = 0$, then π satisfies the property of the theorem.

   Using 2.3, show that if $\mu(\pi) > 0$, it is possible to build a derivation of $\vdash \Gamma$ that is strictly smaller than π wrt μ. Conclude.
4. (Herbrand for Σ_0^1) Consider a provable sequent of the form $\vdash \exists x A(x)$ where A is quantifier free. Show that there exists $t_1, \ldots, t_n$ such that $\vdash A(t_1), \ldots, A(t_n)$ is derivable.

   Thus, even if classical logic does not enjoy the witness property, we can still say something about a classical proof of an existential.

Exercice 4 *Herbrand's theorem*

We consider a formula A of the form $\exists x \forall y B(x, y)$ where B is quantifier-free. We consider a language $\mathscr{L}'$ which is $\mathscr{L}$ enriched with a symbol of function f with arity one.

We write $A^H = \exists x B(x, f(x))$.

1. Using Gödel's completeness theorem for first-order logic and the axiom of Choice, prove that $\vdash A$ if and only if $\vdash A^H$.

By 3.3, then we know that $\vdash A$ is provable if and only if there are terms $t_1, \ldots, t_n$ in $\mathcal{L}'$, such that $\vdash B(t_1, f(t_1)), \ldots, B(t_n, f(t_n))$ where t_i is closed. Such a disjunction is called a Herbrand disjunction. The magic of Herbrand's theorem is that it gives an effective way to compute an Herbrand disjunction.

Theorem 2 ***Herbrand.*** — *A is provable if and only if there exists a Herbrand disjunction for A.*

Note that the theorem extends to arbitrary prenex formula with any number of $\exists\forall$ blocks.

2. Extract a Herbrand disjunction from the proof found in question 2.2.
3. Assume we have a proof π of $\vdash A$. By the midsequent theorem we can assume it starts with contraction, in π there is a subsequent $\vdash A, \ldots, A$ with A repeated n times. Extract a Herbrand disjunction from this proof.