TD10: Révisions

simon.castellan@ens-lyon.fr

Exercise 1 (Warming up)

1. Inhabit the following types of the simply-typed λ-calculus:
   - $r : X \to (X \to R) \to R$
   - $m : (A \to B) \to ((A \to R) \to R) \to (B \to R) \to R$
   - $s : A \times ((B \to R) \to R) \to (A \times B \to R) \to R$
   - $s' : ((A \to R) \to R) \times B \to (A \times B \to R) \to R$
   - $j : ((((X \to R) \to R) \to R) \to R) \to (X \to R) \to R$
2. Do we have the following $\beta\eta$-equalities ?
   - $x : (X \to R) \to R \vdash j(rx) = x = j(mrx) : (X \to R) \to R$
   - $x : ((A \to R) \to R) \times ((B \to R) \to R) \vdash j(ms'(sx)) = j(ms(s'x)) : (A \times B \to R) \to R$
3. The type of lists of type A in systeme F is:

$$\forall X, X \to ((A \to X) \to X) \to X$$

   Every closed term of this type is a finite lists of A.

   How to build a type for finite and infinite lists? (Hint: use existential quantification)

Exercise 2

Show that every normal form is typeable in the Curry-style System F.

Exercise 3 (Kripke models)

A Kripke model is a preorder $E, \leq)$ (ie. $\leq$ is a binary relation on E which is transitive and reflexive.) along with a binary relation $\models \subseteq E \times V$ specifying which propositional variables hold at which nodes, satisfying that if $w \models X$ and $w \leq w'$ then $w' \models X$ for $X \in V$.

Given a Kripke model $(E, \leq)$ and a point $w \in E$, we define $(E, w) \models A$ for a proposition A by induction on A:

- $(E, w) \models X$ if $w \models X$.
- $(E, w) \models \bot$ is never true
- $(E, w) \models \top$ is always true
- $(E, w) \models A \Rightarrow B$ if for all $w' \geq w$, if $(E, w') \models A$ then $(E, w') \models B$
- $(E, w) \models A \vee B$ if $(E, w) \models A$ or $(E, w) \models B$
- $(E, w) \models A \wedge B$ if $(E, w) \models A$ and $(E, w) \models B$

1. Show that if $\vdash A$ is derivable in NJ, then for all Kriple model $(E, \leq)$ and $w \in E$, we have $(E, w) \models A$
2. Deduce that the excluded middle is not provable in NJ.
3. Show that if a proposition is true in a Kripke model, then it is true in a Kripke model which is ordered.
4. Show that if a proposition is true in a Kripke model, then it is true in a Kripke model which is a tree.
5. Show that if a proposition A is true in a Kripke model, then it is true in a Kriple model which is finite and whose size is bounded by a function of the size of A.
6. Assuming *completeness*: if all propositions true in all Kripke models are provable in NJ, show that provability in NJ is decidable.

Exercise 4 (Undecidability of inhabitation of F_ω.)
The goal of this exercice is to prove that the problem: "Given a well-formed proposition A in the context Γ, is there a term such that $\Gamma \vdash t : A$.
For that, we will reduce convertibility in combinatory logic to this problem. Consider the language of terms with two constants S and K and with a binary application $\cdot$ along with the following equations:

$$S \cdot x \cdot y \cdot z = x \cdot z \cdot (y \cdot z) \qquad K \cdot x \cdot y = x$$

Theorem. Given two terms in this language, deciding whether they are equal is undecidable.

1. Show how to build a context Γ containing type constructors so that for each term t there is a proposition $[\![t]\!]$ such that $\Gamma \vdash [\![t]\!] : *$.
2. Show how to build a context Δ such that if $t = u$ $\Gamma, \Delta \vdash [\![t]\!] \to [\![u]\!]$ is inhabited in F_ω.
3. Explain informally how to prove the converse.
4. Deduce the result.