TD2: Intuitionistic logic

simon.castellan@ens-lyon.fr

We recall the rules of propositional intuitionic logic. First, the syntax of propositions:

$$A ::= p \in \mathcal{E} \mid \top \mid \bot \mid A \wedge B \mid A \vee B \mid A \Rightarrow B$$

Contexts (notation: Γ) are lists of propositions.
Then we define *intuitionistic natural deduction* (NJ) as the following system of inferences:

$$\frac{}{\Gamma \vdash \top}\ \top\text{-I} \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash A}\ \bot\text{-E} \qquad \frac{}{\Gamma, A \vdash A}\ \text{Axiom} \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge\text{-I} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ \wedge\text{-E(L)}$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\ \wedge\text{-E(R)} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ \vee\text{-I(L)} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ \vee\text{-I(R)} \qquad \frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C}\ \vee\text{-E}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\ \Rightarrow\text{-I} \qquad \frac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}\ \Rightarrow\text{-E}$$

We say that NJ proves A whenever $\vdash A$ is derivable (empty context).

Exercise 1 (Warmup!)
Provide in NJ a proof of the following propositions:

- $(A \Rightarrow A)$
- $(A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$
- $A \wedge B \Rightarrow A \vee B$

Exercise 2 (Contraction and weakening)
Show the following properties:

(Weakening) If $\Delta \vdash B$, then (for all proposition A) $\Delta, A \vdash B$;

(Contraction) If $\Delta, A, A \vdash B$, then $\Delta, A \vdash B$.

Exercise 3 (Not a non-interesting exercise)
We recall that in NJ negation is encoded as follows $\neg A = A \Rightarrow \bot$. Provide a proof of the following propositions in NJ:

- $A \Rightarrow \neg\neg A$
- $\neg(A \wedge \neg A)$
- $\neg\neg\neg A \Rightarrow \neg A$

Exercise 4 (De Morgan laws)
For the following well-known equivalences, decide for each side if it holds or provide an informal argument to refute them.

- $\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$
- $\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$

Exercise 5 (A logic koan: Master Foo and the excluded middle)
Master Foo once was visited by a student in logic asking him about the true meaning of the excluded middle: "how can I know whether something is true or false? I do not know wether I will die an old man?". Master Foo smiled and made the following offer: "I will choose between giving you 1.000 bananas so you shall never go hungry, or if you gave me that many bananas, I will grant you any wish."
The student accepts and Master Foo opts for the latter. The student wishes for a complete and thorough understanding of logic. Master Foo accepts and lets him wander in the depth of the forest. Later he comes back with the bananas he had to steal from a local gang of monkeys.
Master Foo replies: "That is very nice, thank you. Did I tell you that while you were gone I changed my mind about the offer?" and hands him his bananas back.
Upon receiving the bananas, the student was enlightened.

Moral of the story: Prove the proposition $\neg\neg(A \vee \neg A)$ in intuitionistic logic.

Exercise 6 (Stability and decidability)
Recall that a formula is *stable* when $\vdash_{NJ} \neg\neg A \Rightarrow A$ and *decidable* when $\vdash_{NJ} A \vee \neg A$.

1. What is the relationship between stability and decidability?
2. Given a proposition A, construct a proposition $F(A)$ such that $F(A)$ stable implies A decidable.
3. Show that stable formulas are stable under:
   - *implication*: if A and B are stable, then so is $A \Rightarrow B$
   - *conjunction*: if A and B are stable, then so is $A \wedge B$