TD3: A bit of semantics

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Exercise 1 (Cuts in natural deduction)
Given a propositional variable A, write P for the proposition $(A \Rightarrow A) \Rightarrow (A \Rightarrow A)$.

1. How many "canonical" proofs of P is there in NJ?
2. Consider this proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P, P \vdash P}}{\vdash P \Rightarrow P \Rightarrow P} \qquad \dfrac{\dfrac{}{(A \Rightarrow A), A \vdash A}}{\vdash P}}{\vdash P \Rightarrow P} \qquad \dfrac{\dfrac{\dfrac{}{(A \Rightarrow A), A \vdash A \Rightarrow A} \qquad \dfrac{}{(A \Rightarrow A), A \vdash A \Rightarrow A}}{(A \Rightarrow A), A \vdash A}}{\vdash P = (A \Rightarrow A) \Rightarrow (A \Rightarrow A)}}{\vdash P}$$

3. Suggest a reduction rule on derivations to model this
4. Give the corresponding rules for conjunction and disjunction.
5. Show that there is no proof of $A \vee \neg A$ in NJ which is normal form for those reduction rules.

Exercise 2 (Pierce's law)
Show that the two following rules give rise to equivalent extensions of NJ:

$$\textsc{excluded-middle}\ \dfrac{}{\Gamma \vdash A \vee \neg A} \qquad\qquad \textsc{pierce's law}\ \dfrac{}{\Gamma \vdash ((A \Rightarrow B) \Rightarrow A) \Rightarrow A}$$

The resulting system is called NK.

Exercise 3 (Boolean semantics of NK)
A *valuation* is a map $\rho : V \to \{0, 1\}$ from propositional variables to truth values.

1. By induction on a proposition A, define $[\![A]\!]\rho \in \{0, 1\}$ for a valuation ρ.
2. Show that if $\vdash_{NK} A$ then $[\![A]\!]\rho = 1$ for all ρ
3. Show the converse, if $[\![A]\!]\rho = 1$ for all ρ then $\vdash_{NK} A$.

Exercise 4 (Kripke models)
A Kripke model is a preorder $E, \leq)$ (ie. $\leq$ is a binary relation on E which is transitive and reflexive.) along with a binary relation $\models \subseteq E \times V$ specifying which propositional variables hold at which nodes, satisfying that if $w \models X$ and $w \leq w'$ then $w' \models X$ for $X \in V$.
Given a Kripke model $(E, \leq)$ and a point $w \in E$, we define $(E, w) \models A$ for a proposition A by induction on A:

- $(E, w) \models X$ if $w \models X$.
- $(E, w) \models \bot$ is never true
- $(E, w) \models \top$ is always true
- $(E, w) \models A \Rightarrow B$ if for all $w' \geq w$, if $(E, w') \models A$ then $(E, w') \models B$
- $(E, w) \models A \vee B$ if $(E, w) \models A$ or $(E, w) \models B$
- $(E, w) \models A \wedge B$ if $(E, w) \models A$ and $(E, w) \models B$

1. Show that if $\vdash A$ is derivable in NJ, then for all Kriple model $(E, \leq)$ and $w \in E$, we have $(E, w) \models A$
2. Deduce that the excluded middle is not provable in NJ.
3. Show that if a proposition is true in a Kripke model, then it is true in a Kripke model which is totally ordered.
4. Show that if a proposition is true in a Kripke model, then it is true in a Kripke model which is a tree.
5. Show that if a proposition A is true in a Kripke model, then it is true in a Kriple model which is finite and whose size is bounded by a function of the size of A.
6. Assuming *completeness*: if all propositions true in all Kripke models are provable in NJ, show that provability in NJ is decidable.

Exercise 5 (Homework assignement – Glivenko's theorem)
Prove Glivenko's theorem: if $\Gamma \vdash A$ in NK, then $\neg\neg\Gamma \vdash \neg\neg A$ in NJ where $\neg\neg\Gamma = \neg\neg A_1, \ldots, \neg\neg A_n$ if $\Gamma = A_1, \ldots, A_n$.