TD4: Curry-Howard isomorphism

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Note: when defining simply typed λ-term it might be convenient to use the notation $\lambda x^S.\ t$ for abstractions in order to keep track of the types of variables.

Exercise 1 (Warmup)

1. Inhabit the following types:

   (a) $(S \to T) \to (T \to U) \to S \to U$
   (b) $(S \to T \to U) \to (S \to T) \to S \to U$
   (c) $(S \to T) \to ((S \to U) \to U) \to (T \to U) \to U$

2. Give the most general type for the following terms:

   - $\lambda gf.\ g\ (f\ (\lambda xy.\ x))\ (f\ (\lambda xy.\ y))$
   - $\lambda nfx.\ n\ f\ (f\ x)$

3. Give a strongly normalizable term that cannot be simply-typed.

4. Give a grammar for λ-terms in normal forms.
   Hint: Use two syntactic classes.

Exercise 2 (The negation of Pierce's law/Excluded middle)
Using the simply-typed λ-calculus, prove that there is no proof of Pierce's law in intuitionistic natural deduction.

Exercise 3 (Products in the $\boldsymbol{\lambda}$-calculus)
The goal of this exercise is to extend the correspondance seen during the lectures between the arrow fragment of NJ and the simply typed λ-calculus to products.

1. Explain how to extend the grammar for the pure (untyped) λ-calculus with products.r

2. Give the typing rules to extend the simply-typed λ-calculus with products.

3. Give the new reduction rules for the products. What do they correspond to on proof of NJ?

4. Explain how to update the normalization proof.

5. What would be a η-rule for products? (akin to $\lambda x.\ t\ x = t : A \to B$ if x does not appear in t).

Exercise 4 (Jusqu'à combien peut-on compter en $\boldsymbol{\lambda}$-calcul)
In this exercise we work with the λ-calculus with products. We recall that the church encoding of a natural number is $\overline{n} = \lambda fx.\ f^n\ x$.

1. What is the natural type N for the Church encoding of natural numbers?

2. Show that if $\vdash t : N$, t is β-equal to the church encoding of a natural number.

3. Call a function $f : \mathbb{N}^k \to \mathbb{N}$ representable when there exists a term $\vdash t : N^k \to N$ when $t(\overline{n_1}, \dots, \overline{n_k})$ normalizes to $\overline{f(n_1, \dots, n_k)}$.

   Show that the class of representable function is closed under composition, and contains constant functions, projections, addition, multiplication and the conditional defined by:

$$\text{cond}(n_1, n_2, n_3) = \begin{cases} n_2 & \text{if } n_1 = 0 \\ n_3 & \text{otherwise} \end{cases}$$

4. Let M be a λ-term such that $f : S \to S, a_1 : N, \dots, a_p : N, x_1 : S, \dots, x_m : S \vdash M : S$. Show that, by induction on the normal form of M that there exists a polynomial $P(n_1, \dots, n_p)$ and $i \le m$ such that

$$M[\overline{n_1}/a_1, \dots, \overline{n_p}/a_p] =_\beta f^{P(n_1, \dots, n_p)} x_i$$

   for all tuple of **non-zero** integers $(n_1, \dots, n_p)$.

5. Deduce that the class of representable function is the smallest class of function closed under composition and containing constant functions, projections, addition, multiplication and the conditional.