TD6: System F (1)

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Exercise 1 (Warming up)

1. Inhabit the following types of System F
   (a) $\forall X.(X \to X)$
   (b) $\forall X.(\forall Y.Y) \to X$.
   (c) $\forall Z \forall Y.(\forall X.(X \to Z)) \to Y \to Z$
   (d) $\forall Z.Z \to (\forall Y.(\forall X.X \to Y) \to Y)$
2. Can you type $\Omega = (\lambda x.\ x\ x)\ (\lambda x.\ x\ x)$ (modulo addition of annotations) in system F?
3. Show that the reduction relation $(\Lambda X.t)T \to t[T/X]$ is strongly-normalizing.
4. Given a type A, give a type $t(A)$ of binary trees whose leaves are labelled by A
5. Recall the type of list of type A. How to modify this definition to talk about heterogenous lists?
6. Show how to encode $\top, \bot, A \wedge B, A \vee B$ as types of system F.
7. Show that modulo adding annotations the usual left projection can be typed with $\forall A \forall B.A \wedge B \to A$.

Exercise 2 (Representation in System F)
Recall the type $Nat = \forall X.X \to (X \to X) \to X$.

1. Explain informally why this type is more expressive that what we have in the simply-typed λ-calculus.
2. Show that all primitive recursive function are definable in system F.

   [Primitive recursion: given f, a k-ary primitive recursive function, and g, a $(k+2)$-ary primitive recursive function, the $(k+1)$-ary function h is defined as the primitive recursion of f and g, i.e. the function h is primitive recursive when

$$h(0, x_1, \ldots, x_k) = f(x_1, \ldots, x_k)$$
$$h(S(y), x_1, \ldots, x_k) = g(y, h(y, x_1, \ldots, x_k), x_1, \ldots, x_k)\,.$$

   where a k-ary function is a function $\mathbb{N}^k \to \mathbb{N}$.]
3. Show that the ackermann function is defined: $\text{ack}(0, n) = n + 1$, $\text{ack}(m, 0) = \text{ack}(m - 1, 1)$ and $\text{ack}(m, n) = \text{ack}(m - 1, \text{ack}(m, n - 1))$

Exercise 3 (Meta-theory of F)

1. State (without proving) the substitution lemmas for System F.
2. Using those lemma, show that subject reduction holds: if $\Gamma \vdash t : T$ and $t \to u$ then $\Gamma \vdash u : T$.
3. Show local confluence of System F: if $\Gamma \vdash t : T$ and $t \to u$, $t \to v$ then there exists w such that $u \to^* w$ and $v \to^* w$.