TD6: System F (2)

simon.castellan@ens-lyon.fr

Exercise 1 (Warming up)

1. Show how to encode $\top, \bot, A \wedge B, A \vee B$ as types of system F. Deduce from a NJ formula A, a system F type $\overline{A}$ such that $\vdash A$ implies that there exists $t : \overline{A}$.
2. State (without proving) the substitution lemmas for System F (à la Church).
3. Using those lemma, show that subject reduction holds (in the system à la Church): if $\Gamma \vdash t : T$ and $t \to u$ then $\Gamma \vdash u : T$.

Exercise 2 (Representation in System F)
Recall the type $Nat = \forall X.X \to (X \to X) \to X$.

1. Explain informally why this type is more expressive that what we have in the simply-typed λ-calculus.
2. Show that all primitive recursive function are definable in system F.

   [Primitive recursion: given f, a k-ary primitive recursive function, and g, a $(k+2)$-ary primitive recursive function, the $(k+1)$-ary function h is defined as the primitive recursion of f and g, i.e. the function h is primitive recursive when

   $$h(0, x_1, \ldots, x_k) = f(x_1, \ldots, x_k)$$
   $$h(S(y), x_1, \ldots, x_k) = g(y, h(y, x_1, \ldots, x_k), x_1, \ldots, x_k)\,.$$

   where a k-ary function is a function $\mathbb{N}^k \to \mathbb{N}$.]
3. Show that the ackermann function is defined: $\text{ack}(0, n) = n + 1$, $\text{ack}(m, 0) = \text{ack}(m - 1, 1)$ and $\text{ack}(m, n) = \text{ack}(m - 1, \text{ack}(m, n - 1))$

Exercise 3
Show that a term $\vdash t : A$ of System F à la Church is strongly-normalizable if and only if its erasure is strongly-normalizable.