ENS DE LYON

**TD de Sémantique et Vérification**

# IV– Topology

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Remember that we define the distance on infinite words over the alphabet $\Sigma$ as follows:

$$d : \Sigma^\omega \times \Sigma^\omega \to \mathbb{R}_+$$
$$d(\sigma_1, \sigma_2) = \begin{cases} 0 & \text{if } \sigma_1 = \sigma_2 \\ 2^{-n} & \text{if } n = \min\{k \mid \sigma(k) \neq \sigma(k)\} \end{cases}$$

$\Sigma^\omega$ equipped with this distance is a metric space.

**Exercise 1.**

Show that the metric space $\Sigma^\omega$ is Cauchy-complete.

Remember that $E$ is Cauchy-complete when every Cauchy-sequence is convergent. A cauchy-sequence is a sequence $(e_n \in E)_{n\in\mathbb{N}}$ such that for all $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that for all $m \geq n$ and $k \in \mathbb{N}$, $d(e_m, e_{m+k}) < \epsilon$.

**Exercise 2.**

Remember that a topological space is **compact** when every covering admits a finite subcovering. We consider $E$ a topological space.

1. Let $X \subseteq E$. Show that $e \in cl(X)$ iff for all open set $O$ containing $e$, $X \cap O \neq \emptyset$.
2. Show that for every distinct word $\sigma, \tau \in \Sigma^\omega$, there exists disjoint open sets $U, V$ such that $\sigma \in U$ and $\tau \in V$.
3. Show that if $E$ is compact, every sequence $(C_n)$ of decreasing (for the inclusion) of non-empty closed subset has a non-empty intersection.
4. Show that if $E$ is compact, any closed set of $E$ is also compact (for the induced topology).
5. Show that if $E$ is compact, then $E$ is sequentially compact: every sequence $(\sigma_n)_{n\in\mathbb{N}}$ has a convergent subsequence.
6. Show that since $\Sigma$ is finite, $\Sigma^\omega$ is compact.
7. Show the converse: every compact subset is also closed. (use 2. and 3.)

**Exercise 3.**

Two LT properties $P, P'$ are equivalent ($P \cong P'$) when $pref(P) = pref(P')$. Prove or disprove: $P \cong P'$ if and only if $cl(P) = cl(P')$. Can you give a topological interpretation?