ENS DE LYON

TD de Sémantique et Vérification

VII– Linear Temporal Logic

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Derived operators:

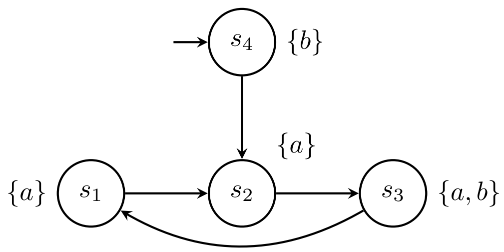
$$\Diamond\varphi := \top \,\mathcal{U}\, \varphi \qquad \varphi\,\mathcal{W}\,\psi := (\varphi\,\mathcal{U}\,\psi) \vee \Box\varphi$$
$$\Box\varphi := \neg\Diamond\neg\varphi \qquad \varphi\,\mathcal{R}\,\psi := \neg(\neg\varphi\,\mathcal{U}\,\neg\psi)$$

Logical equivalence:

$$\varphi \equiv \psi \quad \text{if and only if} \quad \forall\sigma \in \Sigma^\omega, \sigma \models \varphi \Leftrightarrow \sigma \models \psi$$

Exercise 1.

Consider the following transition system over the set of atomic propositions $\{a, b\}$:



Indicate for each of the following LTL formulae the set of states for which these formulae are fulfilled:

(a) $\bigcirc a$
(b) $\bigcirc\bigcirc\bigcirc a$
(c) $\Box b$
(d) $\Box\Diamond a$
(e) $\Box(b\,\mathcal{U}\,a)$
(f) $\Diamond(a\,\mathcal{U}\,b)$

Exercise 2.

Suppose we have two users, *Peter* and *Betsy*, and a single printer device *Printer*. Both users perform several tasks, and every now and then they want to print their results on the *Printer*. Since there is only a single printer, only one user can print a job at a time. Suppose we have the following atomic propositions for *Peter* at our disposal:

- *Peter.request*: indicates that *Peter* requests usage of the printer;
- *Peter.use*: indicates that *Peter* uses the printer;
- *Peter.release*: indicates that *Peter* releases the printer.

For *Betsy*, similar predicates are defined. Specify in LTL the following properties:

(a) Mutual exclusion, i.e., only one user at a time can use the printer.
(b) Finite time of usage, i.e., a user can print only for a finite amount of time.
(c) Absence of individual starvation, i.e., if a user wants to print something, he/she eventually is able to do so.

(d) Absence of blocking, i.e., a user will always request to use the printer.

(e) Alternating access, i.e., users must strictly alternate in printing.

Some equivalence rules for LTL

duality law

$$\neg \bigcirc \varphi \equiv \bigcirc \neg \varphi \quad (1)$$
$$\neg \Diamond \varphi \equiv \Box \neg \varphi \quad (2)$$
$$\neg \Box \varphi \equiv \Diamond \neg \varphi \quad (3)$$

idempotency law

$$\Diamond\Diamond\varphi \equiv \Diamond\varphi \quad (4)$$
$$\Box\Box\varphi \equiv \Box\varphi \quad (5)$$
$$\varphi\,\mathcal{U}\,(\varphi\,\mathcal{U}\,\psi) \equiv \varphi\,\mathcal{U}\,\psi \quad (6)$$
$$(\varphi\,\mathcal{U}\,\psi)\,\mathcal{U}\,\psi \equiv \varphi\,\mathcal{U}\,\psi \quad (7)$$

absorption law

$$\Diamond\Box\Diamond\varphi \equiv \Box\Diamond\varphi \quad (8)$$
$$\Box\Diamond\Box\varphi \equiv \Diamond\Box\varphi \quad (9)$$

expansion law

$$\varphi\,\mathcal{U}\,\psi \equiv \psi \vee (\varphi \wedge \bigcirc(\varphi\,\mathcal{U}\,\psi)) \quad (10)$$
$$\Diamond\psi \equiv \psi \vee \bigcirc\Diamond\psi \quad (11)$$
$$\Box\psi \equiv \psi \wedge \bigcirc\Box\psi \quad (12)$$

distributive law

$$\bigcirc(\varphi\,\mathcal{U}\,\psi) \equiv (\bigcirc\varphi)\,\mathcal{U}\,(\bigcirc\psi) \quad (13)$$
$$\Diamond(\varphi \vee \psi) \equiv \Diamond\varphi \vee \Diamond\psi \quad (14)$$
$$\Box(\varphi \wedge \psi) \equiv \Box\varphi \wedge \Box\psi \quad (15)$$

Exercise 3.

Prove that for any formulae φ_1, φ_2, and φ_3, if for every trace σ, $\sigma \models \neg\varphi_2$ entails $\sigma \models \varphi_3$, then the following equivalences hold:

$$\varphi_1\,\mathcal{U}\,\varphi_2 \equiv (\varphi_1 \wedge \varphi_3)\,\mathcal{U}\,\varphi_2 \qquad\qquad \varphi_1\,\mathcal{W}\,\varphi_2 \equiv (\varphi_1 \wedge \varphi_3)\,\mathcal{W}\,\varphi_2$$

Exercise 4.

Which of the following equivalences are correct? Prove the equivalence or provide a counterexample that illustrates that the formula on the left and the formula on the right are not equivalent.

(a) $\Box\varphi \rightarrow \Diamond\psi \equiv \varphi\,\mathcal{U}\,(\psi \vee \neg\varphi)$

(b) $\Diamond\Box\varphi \rightarrow \Box\Diamond\psi \equiv \Box\,(\varphi\,\mathcal{U}\,(\psi \vee \neg\varphi))$

(c) $\Box\Box\,(\varphi \vee \neg\psi) \equiv \neg\Diamond\,(\neg\varphi \wedge \psi)$

(d) $\Diamond\,(\varphi \wedge \psi) \equiv \Diamond\varphi \wedge \Diamond\psi$

(e) $\Box\varphi \wedge \bigcirc\Diamond\varphi \equiv \Box\varphi$

(f) $\Diamond\varphi \wedge \bigcirc\Box\varphi \equiv \Diamond\varphi$

(g) $\Box\Diamond\varphi \rightarrow \Box\Diamond\psi \equiv \Box\,(\varphi \rightarrow \Diamond\psi)$

(h) $\neg\,(\varphi\,\mathcal{U}\,\psi) \equiv \neg\psi\,\mathcal{W}\,(\neg\varphi \wedge \neg\psi)$

(i) $\bigcirc\Diamond\varphi \equiv \Diamond\bigcirc\varphi$

(j) $(\Diamond\Box\varphi) \wedge (\Diamond\Box\psi) \equiv \Diamond\,(\Box\varphi \wedge \Box\psi)$

Exercise 5.

We consider the release operator $\mathcal{R}$ which was defined by $\varphi\,\mathcal{R}\,\psi := \neg\,(\neg\varphi\,\mathcal{U}\,\neg\psi)$.

1. Prove the expansion law $\varphi\,\mathcal{R}\,\psi \equiv \psi \wedge (\varphi \vee \bigcirc\,(\varphi\,\mathcal{R}\,\psi))$.
2. Prove that $\varphi\,\mathcal{R}\,\psi \equiv (\neg\varphi \wedge \psi)\,\mathcal{W}\,(\varphi \wedge \psi)$.
3. Prove that $\varphi\,\mathcal{W}\,\psi \equiv (\neg\varphi \vee \psi)\,\mathcal{R}\,(\varphi \vee \psi)$.
4. Prove that $\varphi\,\mathcal{U}\,\psi \equiv \neg\,(\neg\varphi\,\mathcal{R}\,\neg\psi)$.